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Particle Aspects of Kink Interactions

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A simple method to compute the interaction between moving solitons in $1+1$ dimensions is presented. Particle aspects of these field configurations are discussed and the interactions of sine-Gordon and ϕ^4 kinks are worked out as examples. The ground-state energy of a kink in a large box is also derived.

1. INTRODUCTION

Topological classification of nonlinear field equations which are of interest in classical as well as quantum field theories can be traced back to Skyrme (1958), Finkelstein (1966), and Coleman (1977).

It is well known that soliton solutions of such equations behave like interacting particles. For example, if free, they move uniformly as localized packets of energy and momentum. They usually retain their identities even after they undergo interactions with other solitons or a background field. Topological charges can be attributed to them which obey a conservation law (see e.g., Rajaraman, 1982). Unlike ordinary conservation laws which follow from underlying symmetries of the field Lagrangian, topological charges come from topological reasons and boundary conditions.

Interesting enough is the fact that a suitable single-field Lagrangian can lead to well-defined particlelike objects (solitons) together with their interactions. Examples also exist for the case of fields with more than one degree of freedom. The basic idea of this kind has been quite fruitful in the description of strong interactions between hadrons. There is even a hope that some type of nonlinear theory will eventually lead to an explanation of elementary particles like quarks or leptons which at present are introduced as sources in most theories (Skyrme, 1988). In this paper we derive a

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simple, interacting particle Lagrangian from a general Lagrangian density which admits topological solitons as their solutions, and apply the idea to find the interaction between moving kinks of the sine-Gordon and ϕ^4 equations. We also manage to calculate the ground-state energy of a kink confined in a large box, when suitable boundary conditions are applied at the box walls.

2. INTERACTING PARTICLE LAGRANGIAN

Consider the Lagrangian density ($c = 1$ is assumed)

$$\mathscr{L} = \frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - V(\phi) \tag{1}$$

with $V(\phi)$ having several degenerate vacua. A well-known example is $V(\phi) = 1 - \cos(\phi)$, which leads to the sine-Gordon equation. We assume the existence of topological solitons $\phi_s(\xi)$ with $\xi = x - vt$ which are solutions of the field equation

$$\frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} = -\frac{\partial V(\phi)}{\partial\phi} \tag{2}$$

derived from the Lagrangian (1). We can defind a current J^μ according to (Ryder, 1985)

$$J^\mu = \frac{1}{2\pi}\epsilon^{\mu\nu}\frac{\partial\phi}{\partial x^\nu} \tag{3}$$

which satisfies the conservation law $\partial J^\mu/\partial x^\mu = 0$, with J^0 the conserved topological charge density. The total charge is obtained via

$$Q = \int_{-\infty}^{+\infty} J^0\, dx = \frac{1}{2\pi}[\phi(+\infty) - \phi(-\infty)] \tag{4}$$

in which $\phi(\infty)$ and $\phi(-\infty)$ correspond to two degenerate vacua of the field ϕ.

We now embed the soliton in a weak background field $\psi(x, t)$ and calculate the total Lagrangian

$$L = \int_{-\infty}^{+\infty}\left[\frac{1}{2}\left(\frac{\partial\phi_s}{\partial t} + \frac{\partial\psi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial\phi_s}{\partial x} + \frac{\partial\psi}{\partial x}\right)^2 - V(\phi_s + \psi)\right]dx \tag{5}$$

ψ is assumed to be of the same nature as ϕ and weak enough such that ϕ_s is not distorted appreciably. Up to the first order in the ψ-interaction terms,

$$L \simeq L_s - 2\pi Q\left.\frac{\partial\psi}{\partial x}\right|_{x_s} - 2\pi Qv\left.\frac{\partial\psi}{\partial t}\right|_{x_s} \tag{6}$$

in which

$$L_s = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left(\frac{\partial \phi_s}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \phi_s}{\partial x}\right)^2 - V(\phi_s) \right] dx$$

$$= \int_{-\infty}^{+\infty} \left(\frac{\partial \phi_s}{\partial t}\right)^2 dx - H_s = -m(1 - v^2)^{1/2} \tag{7}$$

H_s is the total free-soliton Hamiltonian

$$H_s = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left(\frac{\partial \phi_s}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial \phi_s}{\partial x}\right)^2 + V(\phi_s) \right] dx = m\gamma$$

L_s is clearly the Lagrangian of a free, relativistic particle of (classical) rest mass $m \equiv H_s(v = 0)$. $\gamma = 1/(1 - v^2)^{1/2}$ as usual. Note that we have excluded the free background field Lagrangian L_ψ,

$$L_\psi = \int_{-\infty}^{+\infty} \left[\frac{1}{2}\left(\frac{\partial \psi}{\partial t}\right)^2 - \frac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 \right] dx \tag{8}$$

Also note that

$$\int_{-\infty}^{+\infty} \frac{\partial \phi_s}{\partial t}\frac{\partial \psi}{\partial t} dx = \int_{-\infty}^{+\infty} \frac{\partial \xi}{\partial t}\frac{\partial \phi_s}{\partial \xi}\frac{\partial \psi}{\partial t}\frac{\partial x}{\partial \xi} d\xi$$

$$\simeq -v \frac{\partial \psi}{\partial t}\bigg|_{x_s} \int_{-\infty}^{+\infty} \frac{\partial \phi_s}{\partial \xi} d\xi$$

$$\simeq -2\pi Q v \frac{\partial \psi}{\partial t}\bigg|_{x_s} \tag{9}$$

In the same way

$$\int_{-\infty}^{+\infty} \frac{\partial \phi_s}{\partial x}\frac{\partial \psi}{\partial x} dx \simeq 2\pi Q \frac{\partial \psi}{\partial x}\bigg|_{x_s} \tag{10}$$

In deriving (9) and (10), we have assumed that the soliton "size" is so small that across it, $\partial\psi/\partial t$ and $\partial\psi/\partial x$ can be taken as approximately constant and equal to their values at $x = x_s$. One should also note that

$$\int_{-\infty}^{+\infty} V(\phi_s + \psi)\, ds \simeq \int_{-\infty}^{+\infty} V(\phi_s)\, dx + \psi(x_s) \int_{-\infty}^{+\infty} \frac{\partial V(\phi_x)}{\partial \phi_s} dx$$

$$+ \frac{1}{2}\psi^2(x_s) \int_{-\infty}^{+\infty} \frac{\partial^2 V(\phi_s)}{\partial \phi_s^2} dx + \cdots$$

The second term vanishes if $V(\phi)$ is symmetric between the two degenerate vacua at $\phi(\infty)$ and $\phi(-\infty)$. For example, if $V(\phi) = 1 - \cos\phi$ and

$\phi_s = 4\ \text{tg}^{-1} e^x$, then

$$\int_{-\infty}^{+\infty} \frac{\partial V}{\partial \phi}\, dx = \int_{-\infty}^{+\infty} \sin(4 \tan^{-1} e^x)\, dx = 0.$$

The third term is ignorable, because it is of second order in ψ. The particle aspect of the soliton is nicely evident from the Lagrangian (6). Compare—just for the sake of similarity—with the Lagrangian of a relativistic charged particle in an electromagnetic field

$$L = -m(1-v^2)^{1/2} - q\phi + q\mathbf{v} \cdot \mathbf{A}$$

The two interaction terms in the Lagrangian (6) enable us to calculate the force exerted on a moving soliton by another soliton and/or a weak background field. The Lagrangian (6) leads to the following "particle" equation of motion:

$$\frac{dp}{dt} = -2\pi Q \left(\frac{\partial^2 \psi}{\partial x^2} + v \frac{\partial^2 \psi}{\partial x\, \partial t} \right)_{x_s} \tag{11}$$

in which $p = m\gamma v$.

3. INTERACTION OF SINE-GORDON KINKS

Equation (2) with $V(\phi) = 1 - \cos \phi$ gives the well-known sine-Gordon equation which possesses kink (antikink) solutions

$$\phi_s(\xi) = 4 \tan^{-1} e^{\pm \gamma \xi} \tag{12}$$

with topological charges $Q = +1\,(-1)$. Note that we have absorbed the usual constants in the dimensionless variables ϕ and ξ.

Consider an antikink $\phi_{\bar{k}}$ initially at rest at $x = 0$, and a kink ϕ_k moving initially with velocity v at x_s,

$$\phi_{\bar{k}} = 4 \tan^{-1} e^{-x}$$

$$\phi_k = 4 \tan^{-1} e^{\gamma(x - vt - x_s)} \tag{13}$$

In the example we are considering, $m = H_s(v = 0) = 8$, and $\phi_{\bar{k}}$ plays the role of $\psi(x, t)$ at $x = x_s$. The calculation of dp_k/dt is straightforward and we get

$$\frac{dp_k}{dt} = -4\pi \tanh(x_s)\, \text{sech}(x_s) \tag{14}$$

Applying (11) to calculate the force of k on $\bar{k}$ at $x = 0$ and $t = 0$, we get

$$\frac{dp_{\bar{k}}}{dt} = 4\pi \tanh(\gamma x_s)\, \text{sech}(\gamma x_s) \tag{15}$$

We obviously recover Newton's third law for the static case $v = 0$. Note also that the force is attractive and $f \approx 8\pi e^{-x_s}$ for $x_s \gg 1$.

The interaction of kinks in the ϕ^4 model

$$\{\phi_s = \phi_0 \tanh[(\phi_0/2)^{1/2}\gamma(x - vt - x_0)]\}$$

with $V(\phi) = \frac{1}{4}(\phi^2 - \phi_0^2)^2$ can be worked out in the same way. Straightforward calculation yields

$$\frac{dp_k}{dt} = -2\phi_0^4\left(\frac{1+v^2}{1-v^2}\right)\tanh\alpha \operatorname{sech}^2\alpha$$
$$\alpha \equiv (2\phi_0\gamma x_0)^{1/2} \tag{16}$$

for the force exerted by an antikink at $x = x_0$ moving at a velocity $-v$, on a kink at $x = +x_0$ moving at a velocity $+v$. For large interkink distances $(R = 2x_0)$ and in the static $(v = 0)$ case, (19) reduces to

$$\frac{dp_k}{dt} \approx -8\phi_0^4 \exp[-(2\phi_0 R)^{1/2}] \tag{17}$$

which agrees with expressions obtained by others (see, e.g., Perring and Skyrme, 1962; Rajaraman, 1977; and Manton, 1977, 1979). The interkink potential leading to (16) is actually very similar to a one-meson exchange potential which is derived from the corresponding quantum field theory and the one-meson-exchange Born amplitude (Rajaraman, 1982).

4. KINK IN A BOX

In this section we calculate the ground-state energy of a kink in a box with inpenetrable walls at $x = \pm L$. That is, we impose proper boundary conditions at $x = \pm L$.

For a sine-Gordon kink this means $\phi = 2\pi$ at $x = +L$ and $\phi = 0$ at $x = -L$, while for ϕ^4 kinks with $V(\phi) = \frac{1}{4}(\phi^2 - \phi_0^2)^2$ we impose $\phi(\pm L) = \pm\phi_0$. For a static kink at $x = 0$ and a large box $(L \gg 1)$, ϕ is slightly different from a static kink solution:

$$\phi(x) = 4\tan^{-1} e^x + \delta(x) \tag{18}$$

Up to the first order in δ, we have

$$E = m + \int_{-L}^{+L}\left[\frac{\partial\delta}{\partial x}\frac{\partial\phi_s}{\partial x} + \frac{\partial V(\phi_s)}{\partial\phi_s}\delta\right]dx \tag{19}$$

where $m = H_s(v = 0)$ for a free kink as before. Performing one integration by parts, we get for the ground-state energy

$$\delta E \equiv E - m = \delta(x)\frac{d\phi_s}{dx}\bigg|_{-L}^{+L} - \int_{-L}^{+L}\delta(x)\frac{d^2\phi_s}{dx^2}\,dx + \int_{-L}^{+L}\delta(x)\frac{\partial V(\phi)}{\partial\phi}\,dx \tag{20}$$

The second and third terms cancel each other by virtue of the equation $d^2\phi/dx^2 = \partial V(\phi)/\partial\phi$ for ϕ_s. In this way we easily get δE without any need to solve the equation for $\delta(x)$, because $\delta(x)$ at $x = \pm L$ is nothing but $\phi_s(\pm L)$ minus boundary values of ϕ (2π and 0 for sine-Gordon and $\pm\phi_0$ for ϕ^4 kinks). This procedure leads to $\delta E \simeq 32e^{-2L}$ for sine-Gordon kinks and $\delta E \simeq 2\sqrt{2}\,\phi_0^{5/2}\exp[-4(\phi_0/2)^{1/2}L]$ for ϕ^4 kinks.

5. CONCLUSION

A simple method for the calculation of soliton interactions in the weak background field approximation was presented, exploiting similarities with classical particle mechanics and particle interactions. Applicability of the method was examined in the case of sine-Gordon and ϕ^4 fields.

We also calculated the (classical) ground-state energy of kinks in a large box.

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